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Dots for Dummies

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Quantum dots pose an interesting problem in which three complications- disorder, interaction and finite size- come together. I describe progress that can be made by combining Random Matrix Theory (RMT) and the Renormalization Group (RG) to attack the problem.

KEY WORDS: renormalization group; quantum dots, Random Matrix Theory.

1. THE DOT

It is a pleasure to be able to celebrate the birthdays of these two eternal youths, Pierre and Jim. One cannot work in condensed matter for long without running into their numerous contributions. In my case I also had the pleasure of running into Pierre himself fairly regularly over the last decade when he was Yale's Deputy Provost for Sciences. As chairman of Physics, I have locked horns with Pierre on numerous occasions over dollars. He would generally begin by responding to all requests with a No, earning the title Dr. No. This was just Pierre saying "Hello, let us talk." I learnt that his No was a No in the complex plane that could adiabatically be rotated into a Yes. One just had to know how to handle Pierre Pressure. It is also here that I became acquainted with his practice of "scale invariant arguments," the idea being that you should not be able to tell from the intensity of the arguments how much we are arguing about. In fact, as time went by, I noticed counter-intuitive scaling violations. Thus, when I took a request for over a million dollars to him, he would read the requested figure from the right to left and launch a vigorous and intense interrogation on the cents requested. By the time we crossed the decimal point to the units place he would have already begun to lose interest and when we got to the millions, he was ready to agree to anything. This is how Pierre has

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allocated millions for science in general at Yale and for Yale Physics in particular. I was however sworn to secrecy, that I would not tell the world that deep down he was a pussy cat, because he liked this tough in-your-face-Bell-Labs image. But now that he has left Yale, the truth about "deep-pockets" can be revealed.

Anyway Pierre and Jim, I see many more productive years of Physics ahead and wish you all the best.

Now for my subject of quantum dots. I ask you not to take offense at the title, it is just my way of preempting any hostile questions from experts.

For our purposes, the dot is an island of size L (in the nanoscale) within which electrons can live. The boundary of the dot is sufficiently irregular that classical motion is chaotic (at and around the Fermi energy). The dot is otherwise dirt-free and motion within is ballistic. Electrons are allowed to tunnel in and out of the dot and the conductance G is measured as a function of the gate voltage V_g. The challenge is to describe the observed[1] series of peak positions and heights on a statistical basis.[2–4]

The relevant energy scales are Δ, the average single particle level spacing and E_T, the Thouless energy defined by $E_T = \hbar v_F/L$ where v_F is the Fermi velocity. The Thouless energy has a dual significance for us. First if the dot is connected to big fat leads, electrons will cross the dot in a time L/v_F and energy will be uncertain by an amount E_T. Thus

$$g \simeq \frac{E_T}{\Delta} \tag{1}$$

single particle levels will contribute to conductance and g will be the dimensional conductance. (Note that in the experiments we consider the leads are weakly coupled and the levels are sharp.)

The second significance of E_T is that within that band we will assume energy levels and wave functions obey statistics given by RMT.[5] This in turns means two things. First, if we find the exact energy levels ε_α and wavefunction ϕ_α and plot the level spacings within E_T (of say the Fermi energy), the resulting distribution will be indistinguishable from that of a random matrix of the same symmetry. The second RMT result needs some elaboration. Consider a circular Fermi system and a concentric annulus of width E_T. In the bulk, this region contains an infinite number of $\mathbf{k}$ states. If we now go the dot of size L, the best we can do is wave packets centered at some $\mathbf{k}$ and of width $1/L$ in both directions. It is readily verified that we can form g such "Wheel-of-fortune" (WOF) states (as in Fig. 1)) within this annulus.[6] Suppose we expand the g exact eigenstates labeled by an index α in this WOF basis labeled by $\mathbf{k}$ via the functions

$$\langle \mathbf{k}|\alpha\rangle = \phi_\alpha(\mathbf{k}). \tag{2}$$

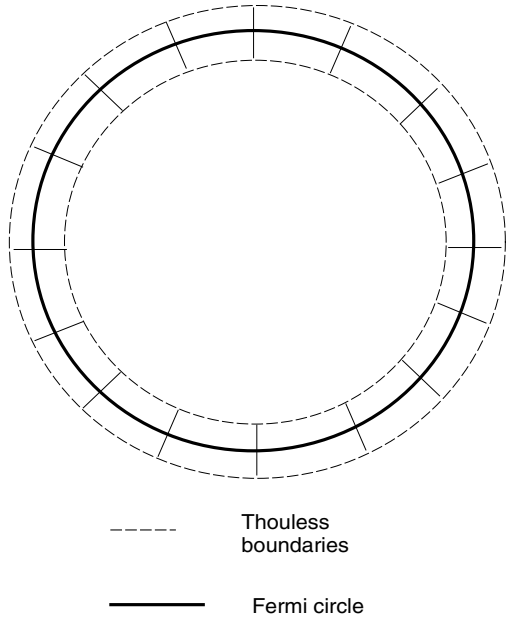

Fig. 1. The Wheel-of-fortune states within a band of energy E_{T} concentric with the Fermi circle. There are roughly g such states of mean momenta $\mathbf{k}$ centered on g equally spaced points on the Fermi circle. The WOF states are obtained by chopping off plane waves of the desired mean momentum at the edges of the dot. These states are nearly orthonormal.

Then a typical RMT assumption made here is that

$$\left\langle \phi_\alpha(\mathbf{k})\phi_\beta(\mathbf{k}')\right\rangle = \frac{\delta_{\mathbf{k}\mathbf{k}'}\delta_{\alpha\beta}}{g} \tag{3}$$

where the $\langle\cdots\rangle$ denote describe an average over an ensemble of similar dots. Note that this is the minimal correlation we must have: in each sample, α and $\mathbf{k}$ label two orthonormal bases, so that if we set $\mathbf{k} = \mathbf{k}'$ and sum over $\mathbf{k}$ we must get $\delta_{\alpha\beta}$, *sample by sample*. (The same goes for setting $\alpha = \beta$ and summing over them to get $\delta_{\mathbf{k}\mathbf{k}'}$.) Similar correlators exist for products of four wavefunctions and these have the form of Wick's theorem.

Before proceeding we address a common question. Is there is any reason to believe that the g exact eigenstates within E_{T} can be expanded in terms of the g WOF states? We have verified[6] the following in a numerical study of a "billiard," or dot. First we manufactured the g states of mean momentum $\mathbf{k}$ by choosing g equally spaced points on the Fermi circle and then chopping off the plane waves of these momenta at the edges of the billiard. (We chose $g = 37$ in our study.) Then we verified that these wavefunctions were orthonormal to an excellent accuracy.

Finally we asked how good a basis these functions formed for expanding the α states within E_T. We found the exact eigenstate at the middle of the Thouless band, i.e, at the Fermi energy, retained more than 99.9% of its norm upon projection to the WOF basis. As we left the center of the band the overlap decreased and dropped to around 50% at the edges. Thus our results are most reliable only deep deep within the Thouless band.

A simple starting hamiltonian for describing dot physics is that of free fermions:

$$H = H_0 = \sum_\alpha \psi_\alpha^\dagger \psi_\alpha \varepsilon_\alpha. \tag{4}$$

If we vary the gate voltage on the dot we expect to see some conductance when the Fermi energy of the electrons in the leads lines up with one of the levels ε_α. Thus we expect the peak spacings to be equal to the typical single particle spacing Δ. The actual value is much larger because when we add an electron to the dot it has electrostatic interactions with the ones already there. This is just the capacitive charging energy $Q^2/2C$. So we need to add a term $u_0 N^2$, where N is the number of electrons on the dot. (Usually this charging energy is subtracted out when data showing conductance versus gate voltage are displayed.) If we take into account the spin of the electrons another term $-J_0 S^2$ is called for, where S is the total spin of the dot. This term reflects the fact that electrons of parallel spins will avoid each other due to the Pauli principle, and the repulsive interaction energy will be lowered at the cost of increased kinetic energy. Thus the hamiltonian at this point takes the form[9–11]

$$H_U = \sum_\alpha \psi_\alpha^\dagger \psi_\alpha \varepsilon_\alpha + u_0 N^2 - J_0 S^2, \tag{5}$$

where the subscript U stands for "universal" and where a third interaction term pertaining to superconducting fluctuations has been dropped.

Some proponents of the universal hamiltonian give the following argument for why no other interactions need be considered. Suppose we take any other familiar interaction and transcribe it to the exact basis. The random wavefunctions ϕ_α will appear and lead to terms with wildly fluctuating signs and phases, with zero ensemble average. Since deviations from the zero average will be down by $1/g$ we can drop them at large g. By contrast the two terms kept commute with H_0 and survive ensemble averaging.

While I am impressed by the success of H_U in explaining a lot of data, this is not so for the accompanying arguments. In particular I think ensemble averages should be performed not on the hamiltonian but calculated observables. I also do not know that a term should be dropped because it is small, since it could prove relevant in the Renormalization Group (RG) sense. I prefer therefore to let the RG

tell us what interactions are important. I turn to a description of an approach based on the RG.

As a prelude, I describe my work[7] on the clean system and in the bulk. Consider a two-dimensional Fermi surface, a circle with Fermi momentum K_F. RG tells us to focus on low energy physics by integrating out high energy degrees of freedom. For bosons and relativistic fermions, this means small momentum. For the fermions however low energy means near the Fermi surface. That is, if I tap the system with a hammer a few fermions near the Fermi surface will jump out, neither those electrons deep in the sea nor the levels far above K_F will be involved. Thus we must eliminate all states but those within a bandwidth Λ of K_F. What theory will we be left with? The story is long, but the following synopsis will suffice for now. Consider the limit $\Lambda \to 0$. Consider an interaction vertex $u(K_1, \ldots, K_4)$ with incoming momenta K_1 and K_2 and outgoing momenta K_3 and K_4. We may take them to lie on the Fermi circle as $\Lambda \to 0$, and thus reduce them to four angles $\theta_1 \ldots \theta_4$. Normally momentum conservation would allow us to consider just three angles. But if the momenta all come from a circle, then θ_3 and θ_4 must equal θ_1 and θ_2 up to a permutation, a result I urge you to verify. Thus u a is function of just θ_1 and θ_2 and by rotational invariance, a function of their difference, θ. This function $u(\theta)$ is none other than Landau's F function,[8] derived here using RG. Often one writes

$$u(\theta) = \sum_m u_m \cos(m\theta) \tag{6}$$

where u_m are the Landau parameters. The corresponding interaction is

$$H_L = \sum_{\theta_1, \theta'} n(\theta) n(\theta') u_m \cos\left(m(\theta - \theta')\right) \tag{7}$$

Note that if only $u_0 \neq 0$, we get the $u_0 N^2$ interaction of H_U. If we include spin, there are spin density-spin density interactions and J_0 corresponds to keeping just the zeorth harmonic. So we need to ask if and when $m > 0$ terms can be ignored.

The first crucial step towards this goal was taken by Murthy and Mathur.[12] Their ideas was as follows.

- **Step 1:** *Use the clean system RG described earlier*[7] *(eliminating momentum states on either side of the Fermi surface) to eliminate all states far from the Fermi surface till one comes down to the Thouless band, that is, within* E_T *of* E_F.
- **Step 2:** *Switch to the exact basis states of the chaotic dot, writing the kinetic and interaction terms in this basis. Run the RG by eliminating exact energy eigenstates within* E_T.

While this looks like a reasonable plan, it is not clear how it is going to be executed since knowledge of the exact eigenfunctions is needed to even write down the Landau interaction in the disordered basis:

$$V_{\alpha\beta\gamma\delta} = \frac{\Delta}{4}\sum_{\mathbf{k}\mathbf{k}'} u(\theta-\theta')\left(\phi^*_\alpha(\mathbf{k})\phi^*_\beta(\mathbf{k}') - \phi^*_\alpha(\mathbf{k}')\phi^*_\beta(\mathbf{k})\right)$$
$$\times(\phi_\gamma(\mathbf{k}')\phi_\delta(\mathbf{k}) - \phi_\gamma(\mathbf{k})\phi_\delta(\mathbf{k}')) \tag{8}$$

(where $\mathbf{k}$ and $\mathbf{k}'$ take g possible values) and to evaluate the flow to one loop. Remarkably it is possible to proceed for the following reason. Let us first assume only one u_m is nonzero. If we write down expression for the one loop flow, four-fold products of the unknown wave functions appear. Now one argues that since many terms enter the sum, there is self-averaging. In other words one can show that if the diagram for any one realization is replaced by the ensemble average, the error is down by a power of $1/g$, and thus ignorable in this large g calculation. Thus no details of the exact wavefunctions are involved in computing the flow! What they find is the remarkable result that the renormalized $V_{\alpha\beta\gamma\delta}$ is itself equivalent to just a single u_m, but of different size. The flow of u_m is given by

$$\frac{du_m}{d\ln\Lambda} = -u_m - cu_m^2 \quad m \neq 0 \tag{9}$$

where c is independent of m and of order unity.

Note that u_0 does not flow and that just as in the BCS flow of the clean system,[7] different m's do not mix to this order. If spin were included J_0 wouldn't flow either.

The flow implies that all positive u_m's flow to zero as do negative ones with $u_m > u^*$, the fixed point of the flow. Thus all points to the right of u^* flow to H_{U}. The universal hamiltonian is thus an RG fixed point with a domain of attraction of order unity.

The work of Murthy and Mathur raises two questions: is the fixed point u^* of order unity to be trusted (coming as it does from a one loop calculation) and what happens to the left of it? This was answered by Murthy and myself.[13] We found that at large g one did not have to rely on RG once one got to down to E_{T} using Step 1. Instead the theory could be solved by saddle point methods employed in $1/N$ expansions with g playing the role of N. The results became exact as $g \to \infty$. I give only a few details here referring you to the above references and ref. 14.

Suppose we have a theory with N flavors defined schematically by a path integral:

$$Z = \int d\psi d\bar{\psi} e^{\bar{\psi}D\psi + u(\bar{\psi}\psi)^2} \tag{10}$$

where D stands for the quadratic kinetic energy term, and the sum over flavors or integral over spacetime is suppressed. If we now introduce a Hubbard-Stratonovic

field σ we may proceed as follows

$$Z = \int d\psi d\bar{\psi} \sigma e^{\bar{\psi}(D+\sigma)\psi - \sigma^2/2u} \tag{11}$$

$$= \int d\sigma e^{NTrln(D+\sigma) - \sigma^2/2u} \tag{12}$$

Using the largeness of N, one does the σ integral by saddle point. If there is any symmetry breaking in σ, this is a reliable way to see it and to to study it.

In our problem, D would stand for the kinetic energy of noninteracting electrons in the dot. Although there were $N = g$ fermion labels (α), they were not related by symmetry to begin with, so that the appearance of an N in front of the Tr Ln was not a given. However if one did the Tr Ln order by order in σ and used self averaging as in the one loop flow, one would find a g^2 in front, playing the role of N.

Since we have a large N theory here it follows that as in all large N theories, the one-loop flow and the new fixed point at strong coupling are parts of the final theory. However the exact location of the critical point cannot be predicted, as pointed out to us by Professor Piet Brower. The reason is that the Landau couplings u_m are defined at a scale E_L much higher than E_{T} (but much smaller than E_{F}) and their flow till we come down to E_{T}, where our analysis begins, is not within the regime we can control. In other words we can locate u^* in terms of what couplings we begin with at E_{T}, but these are the Landau parameters renormalized in a nonuniversal way as we come down from E_L to E_{T}.

What is the nature of the state for $u_m \leq u_m^*$?

In the strong coupling region $\boldsymbol{\sigma}$ acquires an expectation value in the ground state. The dynamics of the fermions is affected by this variable in many ways: quasi-particle widths become broad very quickly above the Fermi energy, the second difference Δ_2 has occasionally very large values and can even be negative,[2] and the system behaves like one with broken time-reversal symmetry if m is odd.[14]

Long ago Pomeranchuk[15] found that if a Landau function of a pure system exceeded a certain value, the fermi surface underwent a shape transformation from a circle to an non-rotationally invariant form. Recently this transition has received a lot of attention[16–17] The transition in question is a disordered version of the same. Details are given in refs. 13 and 14.

Details aside, there is another very interesting point: even if the coupling does not take us over to the strong-coupling phase, we can see vestiges of the critical point u_m^* and associated critical phenomena. This is a general feature of

[2] How can the cost of adding one particle be negative (after removing the charging energy)? The answer is that adding a new particle sometimes lowers the energy of the collective variable which has a life of its own. However, if we turn a blind eye to it and attribute all the energy to the single particle excitations, Δ_2 can be negative.

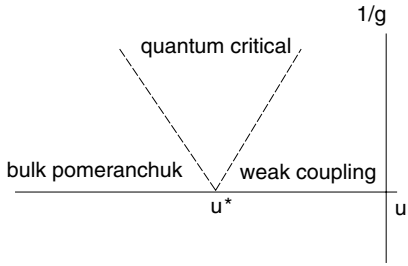

Fig. 2. The generic phase diagram for a second-order quantum phase transition. The horizontal axis represents the coupling constant which can be tuned to take one across the transition. The vertical axis is usually the temperature in bulk quantum systems, but is $1/g$ here, since in our system one of the roles played by g is that of the inverse temperature.

many *quantum critical points*,[18] i.e., points like u_m^* where as a variable in a hamiltonian is varied, the system enters a new phase (in contrast to transitions wherein temperature T is the control parameter).

Figure 2 shows what happens in a generic situation. On the x-axis a variable (u_m in our case) along which the quantum phase transition occurs. Along y is measured a new variable, usually temperature T. Let us consider that case first. If we move from right to left at some value of T, we will first encounter physics of the weak-coupling phase determined by the weak-coupling fixed point at the origin. Then we cross into the *critical fan* (delineated by the V-shaped dotted lines), where the physics is controlled by the quantum critical point. In other words we can tell there is a critical point on the x -axis without actually traversing it. As we move further to the left, we reach the strongly-coupled symmetry-broken phase, with a non-zero order parameter.

It can be shown that in our problem, $1/g^2$ plays the role of T. That is, g^2 stands in front of the effective action for $\boldsymbol{\sigma}$. Here g also enters at a subdominant level inside the action, which makes it hard to predict the exact shape of the critical fan. The bottom line is that we can see the critical point at finite $1/g$. In addition one can also raise temperature or bias voltage to see the critical fan.

Subsequent work has shown, in more familiar examples that Landau interactions, that the general picture depicted here is true in the large g limit: upon adding sufficiently strong interactions the Universal Hamiltonian gives way to other descriptions with broken symmetry.[19]

I mentioned that the critical point u^* (a nonuniversal quantity) cannot be reliably predicted in the large g limit. It now is clear from numerical work that it coincides with the bulk coupling for the Pomeranchuk transition. In other words, when we cross over to the left u^*, the size of the order parameter very rapidly grows from the mesoscopic scale of order E_{T} to something of order the Fermi energy. However the physics in the critical fan as well as the weak coupling side is as described by

our RMT+RG analysis. The strong coupling side has to be reworked from scratch since the Fermi surface assumed in the RG that came down to E_T is has suffered huge deformations (in the scale of E_T).

You can find additional details in my recent lectures in south Africa.[20]

To conclude, it is possible to understand interacting electrons in a quantum dot of irregular shape by combining RMT and RG as long as the dot is large, i.e., $g \to \infty$. (Small dots that have large level spacings behave more like atoms and cannot be handled in the present scheme.) The RG allows us to understand the Universal Hamiltonian as fixed point for a range of couplings, a view I find more satisfactory that those based other considerations. The u versus $1/g$ phase diagram allows us to explore a region not described by the Universal Hamiltonian without having to go to couplings as large as the critical coupling. We should of course remember that at strong coupling there are other possibilities besides the Pomeranchuk phase described here. It will be very interesting if experimentalists could be induced to study strongly interacting dots, a possibility more readily realized in dots than in the bulk since electron density in can be controlled by gates. Stay tuned as we await this.

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